Fourier Series for Periodic Functions

Lecture #8

5CT3,4,6,7

Homework

- Problem (1)
  - Compute the Fourier Series for the periodic functions

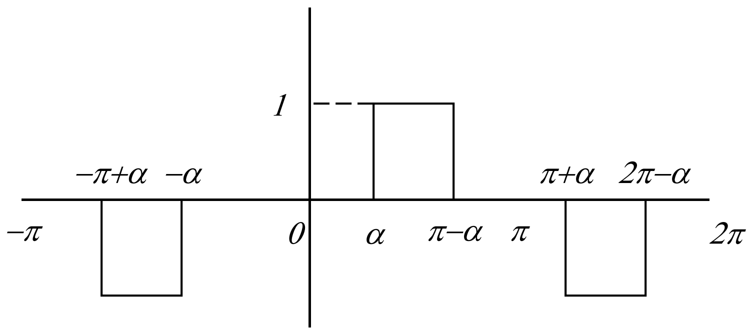
    a) $f(t) = 1$ for $0 < t < \pi$, $f(t)=0$ for $\pi < t < 2\pi$

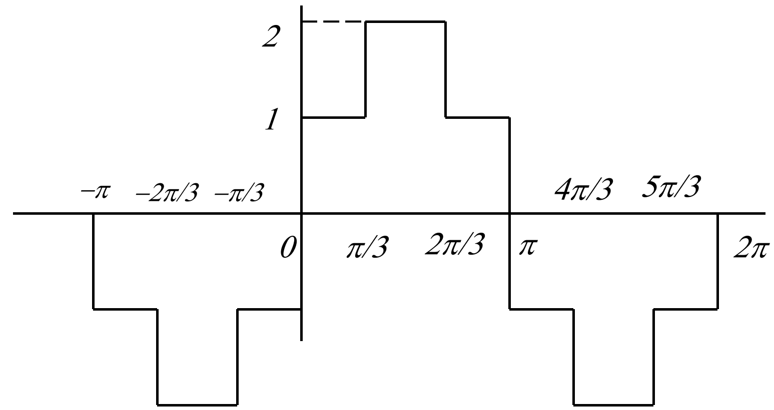
    b) $f(t) = t$ for $0 < t < 3$
- Problem (2)
  - Compute the Fourier series of the following Periodic Functions:
    - $f(t)= t,\ 2n\pi < t < (2n+1)\pi$ for $n \geq 0$

      $= 0,\ (2n+1)\pi < t < (2n+2)\pi$ for $n \geq 0$
    - $f(t)= e^{-t/\pi},\ 2n\pi < t < (2n+2)\pi$ for $n \geq 0$ Use Matlab to plot $f(t)$ using a_k for maximum number of components, N=5,10, 100, and1000. Show your code.
- Problem (3)
  - Problems: 4.1, 4.3
- Problem (4)
  - Using Matlab with its integration function, calculate and plot the spectrum for both the square wave and triangular wave approximation of a periodic ECG. Plot the spectrum in Hz and then plot the ECG time signal using the Fourier spectral coefficients. Submit your code.

Homework

- Problem (4)
  - Deduce the Fourier series for the functions shown (hint: deduce the second one using superposition):





- 5CT.7.1

Homework

- Problem (4)
  - Using Matlab with its integration function, calculate and plot the spectrum for both the square wave and triangular wave approximation of a periodic ECG. Plot the spectrum in Hz and then plot the ECG time signal using the Fourier spectral coefficients. Submit your code.

Homework Answers #1

- Fourier Series
  - Problem (1a)
    - Compute the Fourier Series for the periodic function :

      $f(t) = 1$ for $0<t<\pi$, $f(t)=0$ for $\pi<t<2\pi$

$$a_k = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-jkt}dt = \frac{1}{2\pi}\int_0^{\pi} 1e^{-jkt}dt = (\frac{1}{2\pi})(\frac{1}{-jk})\, e^{-jkt}\Big|_0^{\pi} = \frac{1}{-2\pi kj}(e^{-jk\pi}-1)$$

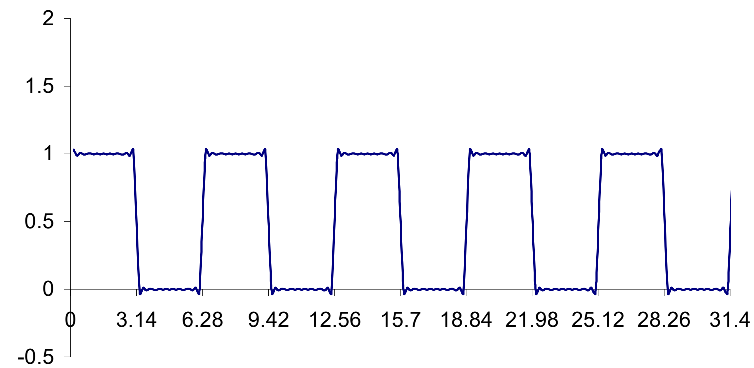
$$= \frac{1}{2\pi kj}(1-e^{-jk\pi}) = \frac{1}{2\pi kj}(1-(-1)^k) =$$

$$= \frac{1}{\pi k}e^{j\pi/2};\ \text{for } k \neq 0; k \text{ odd}$$

$$=0;\ \text{for } k \neq 0; k \text{ even}$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)dt = \frac{1}{2\pi}\int_0^{\pi} 1dt = \frac{1}{2}$$

$$f(t) = \frac{1}{2} + 2\sum_{kodd=1}^{\infty}\frac{1}{\pi k}\cos(kt-\frac{\pi}{2})$$



Homework Answers #2

- Fourier Series
  - Problem (1b)
    - Compute the Fourier Series for the periodic function:

      f(t) = t for *0<t<3*

$$a_k = \frac{1}{3}\int_0^3 f(t)e^{-j2\pi kt/3}dt = \frac{1}{3}\int_0^3 te^{-j2\pi kt/3}dt$$

$$u = t, du = dt$$

$$dv = e^{-j2\pi kt/3}dt, v = \frac{1}{-j2\pi k/3}e^{-j2\pi kt/3}$$

$$\frac{1}{3}\int_0^3 te^{-jkt}dt = \frac{1}{3}[\frac{t}{-j2\pi k/3}e^{-j2\pi kt/3}\Big|_0^3 - \int_0^3 (\frac{1}{-j2\pi k/3})e^{-j2\pi kt/3}dt]$$

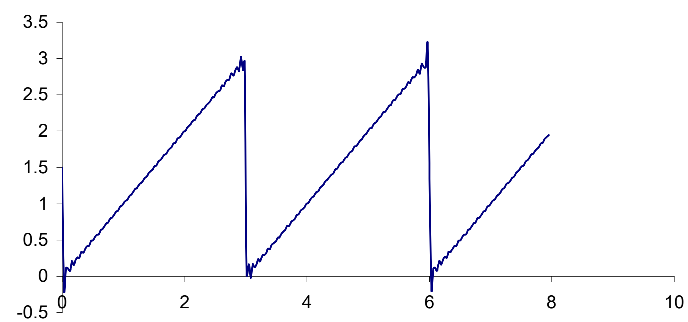
$$= \frac{1}{3}[\frac{1}{-j2\pi k/3}\{3e^{-j2\pi k} - 0\} - (\frac{1}{-j2\pi k/3})(\frac{1}{-j2\pi k/3}\{e^{-j2\pi k} - 1\}$$

since $e^{-j2\pi k} = \cos(2\pi k) + j\sin(2\pi k) = 1 + j0$

$$a_k = \frac{1}{-j2\pi k/3} = j\frac{3}{2\pi k}; k \neq 0$$

$$a_0 = \frac{1}{3}\int_0^3 f(t)dt = \frac{1}{3}\int_0^3 tdt = (\frac{1}{3})\frac{9}{2} = 1.5$$

$$f(t) = 1.5 + 2\sum_1^\infty \frac{3}{2\pi k}\cos(2\pi kt/3 + \pi/2)$$



Homework Answers #3

- Problem (2a)
  - Compute the Fourier series of the following Periodic Functions:
    - $f(t) = t,\ 2n\pi < t < (2n+1)\pi$ for $n \geq 0$

      $= 0,\ (2n+1)\pi < t < (2n+2)\pi$ for $n \geq 0$

$$a_k = \frac{1}{2\pi}\int_0^{\pi} te^{-jkt}dt$$

$$u = t, du = dt$$

$$dv = e^{-jkt}dt, v = \frac{1}{-jk}e^{-jkt}$$

$$\frac{1}{2\pi}\int_0^{\pi} te^{-jkt}dt = \frac{1}{2\pi}[\frac{t}{-jk}e^{-jkt}|_0^{\pi} - \int_0^{\pi}(\frac{1}{-jk})e^{-jkt}dt]$$

$$= \frac{1}{2\pi}[\frac{1}{-jk}\{\pi e^{-j\pi k} - 0\} - (\frac{1}{-jk})(\frac{1}{-jk})\{e^{-j\pi k} - 1\}]$$

Homework Answers #4

- Problem (2a)
  - Compute the Fourier series of the following Periodic Functions:
    - $f(t) = t,\ 2n\pi < t < (2n+1)\pi$ for $n \geq 0$

      $= 0,\ (2n+1)\pi < t < (2n+2)\pi$ for $n \geq 0$

$$a_k = j\frac{e^{-j\pi k}}{2\pi k}[\pi - (\frac{1}{-jk})\{1-e^{j\pi k}\}]$$

$$= \frac{e^{-j\pi k}}{2\pi k}[j\pi + (\frac{1}{k})\{1-e^{j\pi k}\}]$$

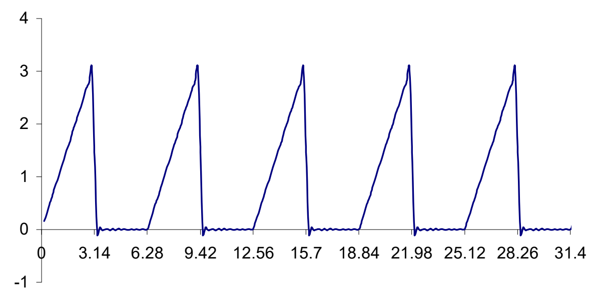
$$= \frac{(-1)^k}{2\pi k}[j\pi + (\frac{1}{k})\{1-(-1)^k\}]$$

since $e^{j\pi k} = \cos(\pi k) + j\sin(\pi k)$

$$= (-1)^k + j0$$

$$a_k = \frac{(-1)^k}{2\pi k}\sqrt{(\frac{1}{k^2})\{1-(-1)^k\}^2 + \pi^2}\angle\tan^{-1}\frac{\pi k}{\{1-(-1)^k\}};k \neq 0$$

$$a_0 = \frac{1}{2\pi}\int_0^{\pi} t dt = \frac{1}{2\pi}\frac{t^2}{2} = (\frac{1}{2\pi})\frac{\pi^2}{2} = \frac{\pi}{4}$$



$$f(t) = \frac{\pi}{4} + 2\sum_{1}^{\infty}\frac{(-1)^k}{2\pi k}\sqrt{(\frac{1}{k^2})\{1-(-1)^k\}^2 + \pi^2}\cos(kt + \tan^{-1}\frac{\pi k}{\{1-(-1)^k\}})$$

Homework Answers #5

- Problem (2b)
  - Compute the Fourier series of the following Periodic Functions:
    - $f(t) = e^{-t/\pi},\ 2n\pi < t < (2n+2)\pi$ for $n \geq 0$

$$a_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-t/\pi} e^{-jkt}\,dt = \frac{1}{2\pi}\int_0^{2\pi} e^{-t(jk+1/\pi)}\,dt$$

$$= \frac{1}{2\pi}\left(\frac{1}{-(jk+1/\pi)}\right) e^{-t(jk+1/\pi)}\Big|_0^{2\pi}$$

$$= -\left(\frac{1}{j2\pi k+2}\right)\{e^{-(j2\pi k+2)} - 1\}$$

$$e^{-(j2\pi k+2)} = e^{-2} e^{-j2\pi k} = e^{-2}$$

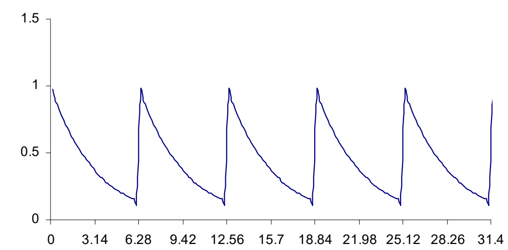
$$a_k = -\left(\frac{1}{j2\pi k+2}\right)\{e^{-2} - 1\}$$

$$= \left(\frac{.865}{j2\pi k+2}\right)$$

$$= \left(\frac{.433}{\sqrt{(\pi k)^2+1}}\right)\angle -\tan^{-1}(\pi k)$$

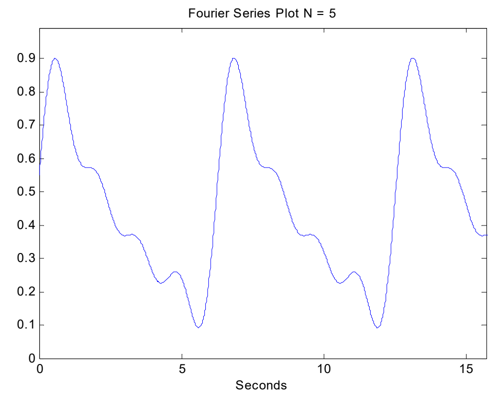
$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} e^{-t/\pi}\,dt = \frac{-\pi}{2\pi} e^{-t/\pi}\Big|_0^{2\pi} = .433$$

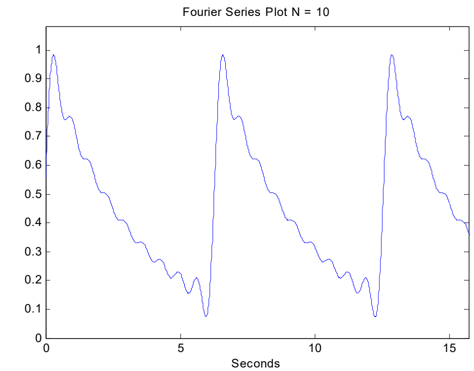
$$f(t) = .433 + 2\sum_1^{\infty} \frac{.433}{\sqrt{(\pi k)^2+1}} \cos[kt - \tan^{-1}(\pi k)]$$

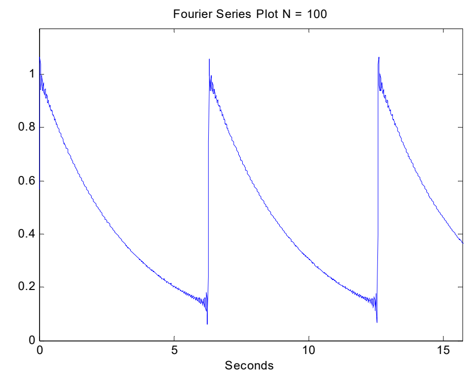


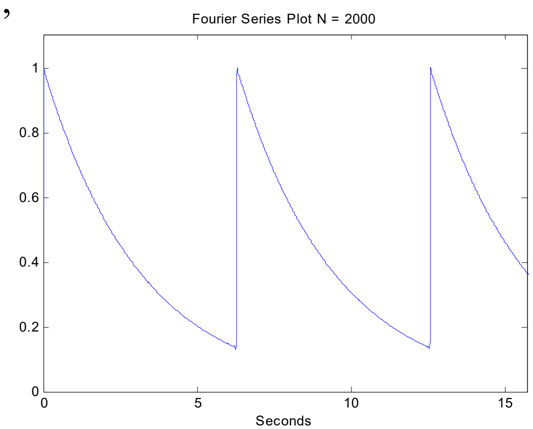
Matlab Code

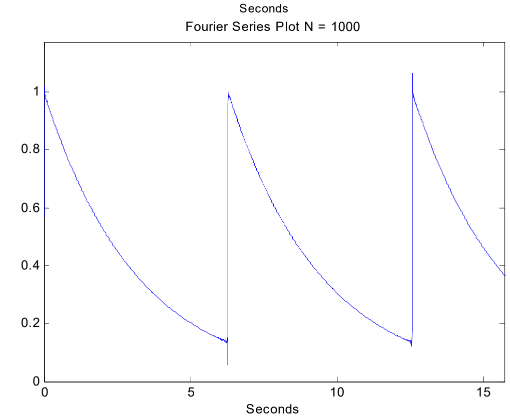
```
clear all;
time=(0:.01:10*pi);
N=2000;
maxtime=length(time);
a=(1-exp(-2))/2;
for i=1:maxtime;
    x=0;
    for j=2:N;
        x=x+2*a/sqrt(1+(pi*(j-1))^2)*cos((j-1)*time(i)-atan(pi*(j-1)));
    end
    y(i)=x+a;
end
plot(time,y);
title(['Fourier Series Plot N = ',num2str(N)]);
xlabel('Seconds');
axis([ time(1) max(time)/2 0 max(y)*1.1]);
```











Homework Answers #6

- Problem (3)
  - Problems: 4.1

Note this waveform has even quarter-wave symmetry: which means odd harmonics and cosine terms:

$$A_{2k-1} = \frac{8}{T}\int_0^{T/4}(1-\frac{4t}{T})\cos[(2k-1)\frac{2\pi}{T}t]dt$$

$$u = (1-\frac{4t}{T}); du = -\frac{4}{T}dt$$

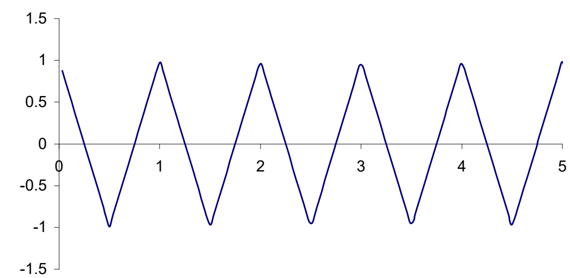
$$dv = \cos[(2k-1)\frac{2\pi}{T}t]dt; v = \frac{T}{(2k-1)2\pi}\sin[(2k-1)\frac{2\pi}{T}t]$$

$$A_{2k-1} = \frac{8V_p}{T}\{\frac{T}{(2k-1)2\pi}(1-\frac{4t}{T})\sin[(2k-1)\frac{2\pi}{T}t]|_0^{T/4} - \int_0^{T/4}(-\frac{4}{T})(\frac{T}{(2k-1)2\pi})\sin[(2k-1)\frac{2\pi}{T}t]dt$$

$$= \frac{8V_p}{T}\{\frac{T}{(2k-1)2\pi}[0-0] - \frac{4}{T}(\frac{T}{(2k-1)2\pi})^2\cos[(2k-1)\frac{2\pi}{T}t]|_0^{T/4}\}$$

$$= \frac{8V_p}{[(2k-1)\pi]^2}[1-0] = \frac{8V_p}{[(2k-1)\pi]^2}$$

$$f(t) = 8V_p\sum_1^{\infty}\frac{1}{[(2k-1)\pi]^2}\cos[(2k-1)\frac{2\pi}{T}t]$$



Homework Answers #6

- Problem (3)
  - Problems: 4.1

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-j\frac{2\pi k}{T}t}dt = \frac{V_p}{T}\{\int_{-T/2}^{0}[1+\frac{4t}{T}]e^{-j\frac{2\pi kt}{T}}dt + \int_{0}^{T/2}[1-\frac{4t}{T}]e^{-j\frac{2\pi kt}{T}}dt\}$$

$$u = 1 \pm \frac{4t}{T}; du = \pm\frac{4}{T}dt$$

$$dv = e^{-j\frac{2\pi k}{T}t}dt; v = \frac{T}{-j2\pi k}e^{-j\frac{2\pi k}{T}t}$$

$$a_k = \frac{V_p}{T}\{[(1+\frac{4t}{T})\frac{T}{-j2\pi k}e^{-j\frac{2\pi k}{T}t}|_{-T/2}^{0} - \int_{-T/2}^{0} + \frac{4}{T}(\frac{T}{-j2\pi k})e^{-j\frac{2\pi k}{T}t}dt] + [(1-\frac{4t}{T})\frac{T}{-j2\pi k}e^{-j\frac{2\pi k}{T}t}|_{0}^{T/2} - \int_{0}^{T/2} - \frac{4}{T}(\frac{T}{-j2\pi k})e^{-j\frac{2\pi k}{T}t}dt]\}$$

$$= \frac{V_p}{T}\{\frac{T}{-j2\pi k}[(1+0)e^{-j\frac{2\pi k}{T}0} - (1-2)e^{j\pi k}] - \frac{4}{T}(\frac{T}{-j2\pi k})^2 e^{-j\frac{2\pi k}{T}t}|_{-T/2}^{0}$$

$$+ \frac{T}{-j2\pi k}[(1-2)e^{-j\pi k} - (1-0)e^{-j\frac{2\pi k}{T}0}] - (-\frac{4}{T})(\frac{T}{-j2\pi k})^2 e^{-j\frac{2\pi k}{T}t}|_{0}^{T/2}\}$$

$$= \frac{V_p}{T}\{\frac{T}{-j2\pi k}(1+e^{j\pi k}) - \frac{4}{T}(\frac{T}{-j2\pi k})^2[1-e^{j\pi k}] + \frac{T}{-j2\pi k}[(-1)(e^{-j\pi k}+1)] - (-\frac{4}{T})(\frac{T}{-j2\pi k})^2(e^{-j\pi k}-1)\}$$

$$= \frac{V_p}{T}\{\frac{T}{-j2\pi k}(1+e^{j\pi k}) - \frac{4}{T}(\frac{T}{-j2\pi k})^2[1-e^{j\pi k}] + \frac{T}{-j2\pi k}[(-1)(e^{-j\pi k}+1)] - (-\frac{4}{T})(\frac{T}{-j2\pi k})^2(e^{-j\pi k}-1)\}$$

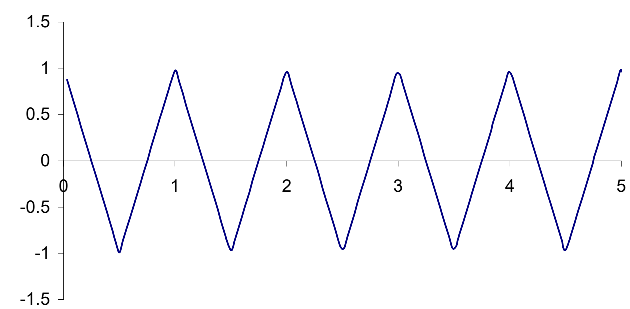
$$Note: e^{j\pi k} = \cos\pi k + j0 = e^{-j\pi k} = \cos\pi k - j0 = (-1)^k$$

$$= \frac{V_p}{T}\{-\frac{4}{T}(\frac{T}{-j2\pi k})^2[1-e^{j\pi k}] + -(-\frac{4}{T})(\frac{T}{-j2\pi k})^2(e^{-j\pi k}-1)\}$$

$$= \frac{V_p}{T}\{\frac{1}{T}(\frac{T}{\pi k})^2[1-e^{j\pi k}] + (\frac{1}{T})(\frac{T}{\pi k})^2[1-e^{-j\pi k}]\}$$

$$= 2V_p(\frac{1}{\pi k})^2[1-(-1)^k]$$

$$= 4V_p(\frac{1}{\pi k})^2 \text{ for } k \text{ odd; 0 for } k \text{ even; } f(t) = 8V_p\sum_{k\ \text{ODD}}^{\infty}(\frac{1}{\pi k})^2\cos\pi k$$



Homework Answer #7

- Problem (4a)
  - Problems: 4.3

$$x(t) = x(t+T_o) \qquad -T_o/2 \le t \le T_o/2$$

$$x(t) = \begin{cases} 1 & \text{for } |t| < t_c = T_o/4 \\ 0 & \text{for } t_c = T_o/4 < |t| \le T_o/2 \end{cases}$$

$$t_c < T_o/2$$

$$a_o = \frac{1}{T_o}\int_{-T_o/2}^{T_o/2} x(t)dt = \frac{1}{T_o}\int_{-t_c}^{t_c} 1dt = \frac{2t_c}{T_o}$$

$$a_k = \frac{1}{T_o}\int_{-T_o/2}^{T_o/2} x(t)e^{-j(2\pi/T_o)kt}dt = \frac{1}{T_o}\int_{-t_c}^{t_c} e^{-j(2\pi/T_o)kt}dt$$

$$= \frac{1}{T_o(-j(2\pi/T_o)k)}e^{-j(2\pi/T_o)kt}\Big|_{-t_c}^{t_c} = \frac{1}{T_o - j(2\pi/T_o)k}[e^{-j(2\pi/T_o)kt_c} - e^{j(2\pi/T_o)kt_c}]$$

$$= \frac{2t_c}{T_o(2\pi/T_o)kt_c}[\frac{e^{j(2\pi/T_o)kt_c} - e^{-j(2\pi/T_o)kt_c}}{2j}] = \frac{2t_c}{T_o}\frac{\sin[(2\pi/T_o)kt_c]}{(2\pi/T_o)kt_c}$$

$$= \frac{2t_c}{T_o}\text{sinc}[(2\pi/T_o)kt_c] = \frac{2T_o/4}{T_o}\text{sinc}[(2\pi/T_o)kT_o/4] = \frac{1}{2}\text{sinc}[(\pi/2)k]$$

Homework Answer #7

- Problem (4a)
  - Problems: 4.3

$$x(t) = x(t + T_o) \qquad -T_o/2 \le t \le T_o/2$$

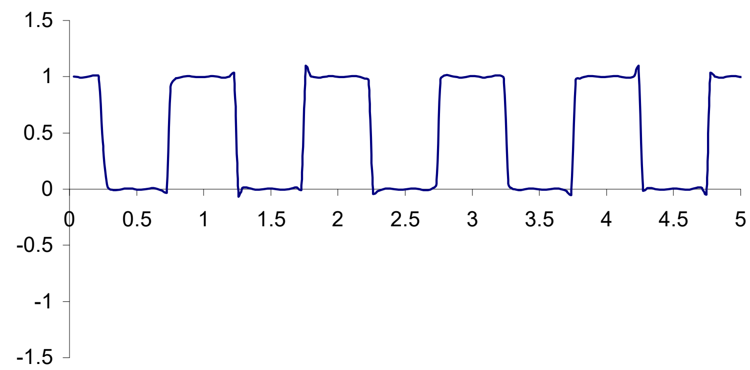
$$x(t) = \begin{cases} 1 & \text{for } |t| < T_o/4 \\ 0 & \text{for } T_o/4 < |t| \le T_o/2 \end{cases}$$

$$T_o = 1$$

$$a_o = \frac{1}{T_o}\int_{-T_o/2}^{T_o/2} x(t)dt = \frac{1}{T_o}\int_{-T_o/4}^{T_o/4} 1dt = \frac{1}{2}$$

$$a_k = \frac{1}{2}\text{sinc}[(\pi/2)k]$$

$$x(t) = \frac{1}{2} + 2\sum_{k=1}^{\infty}\frac{1}{2}\text{sinc}[(\pi/2)k]\cos(2\pi kt) = \frac{1}{2} + \sum_{k=1}^{\infty}\frac{\sin[(\pi/2)k]}{(\pi/2)k}\cos(2\pi kt)$$



Homework Answer #7

- Problem (4a) ALTERNATIVE SOLUTION
  - Problems: 4.3 Even symmetry only cosines.

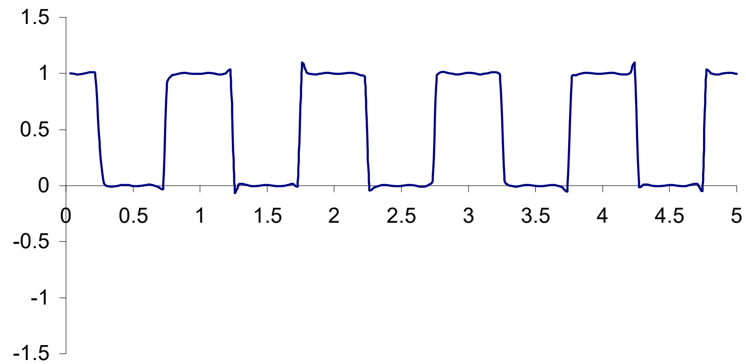
$$A_k = \frac{V_0 4}{T}\int_0^{T/4} \cos[\frac{2\pi k}{T}t]dt$$

$$= \frac{V_0 4}{T}\frac{T}{2\pi k}\sin[\frac{2\pi k}{T}t]|_0^{T/4}$$

$$A_k = \frac{V_0}{\frac{\pi k}{2}}\sin[\frac{\pi k}{2}]$$

$$A_0 = \frac{V_0 2}{T}\int_0^{T/4} dt + \frac{0}{T}\int_{T/4}^{T/2} dt = \frac{V_0}{2}$$

$$f(t) = \frac{V_0}{2} + V_0 \sum_1^{\infty} \frac{\sin[\frac{\pi k}{2}]}{[\frac{\pi k}{2}]}\cos[\frac{2\pi k}{T}t]$$



Homework Answer #7

- Problem (4a)
  - Problems: 4.3 Even symmetry only cosines.

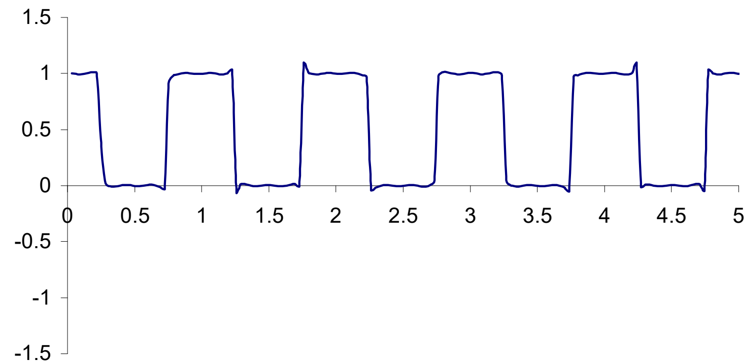
$$a_k = \frac{V_0}{T}\int_{-T/2}^{T/2} e^{-j\frac{2\pi kt}{T}}\,dt = \frac{V_0}{T}\int_{-T/4}^{T/4} e^{-j\frac{2\pi kt}{T}}\,dt$$

$$= \frac{V_0}{T}\left(\frac{T}{-j2\pi k}\right)e^{-j\frac{2\pi kt}{T}}\,]\Big|_{-T/4}^{T/4}$$

$$= \frac{V_0}{T}\left(\frac{T}{-j2\pi k}\right)\left(e^{-j\frac{\pi k}{2}} - e^{j\frac{\pi k}{2}}\right)] = \frac{V_0}{2}\,\frac{\sin(\pi k/2)}{\pi k/2}$$

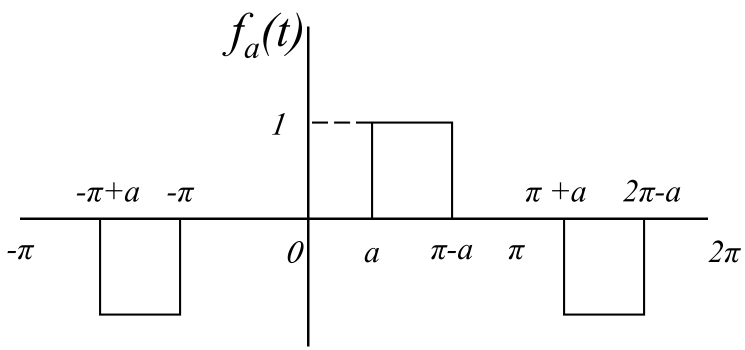
$$a_0 = \frac{V_0}{T}\int_{-T/2}^{T/4} dt = \frac{V_0}{T}[T/4 - (-T/4)] = \frac{V_0}{2}$$

$$f(t) = \frac{V_0}{2} + V_0\sum_{1}^{\infty}\frac{\sin[\frac{\pi k}{2}]}{[\frac{\pi k}{2}]}\cos[\frac{2\pi k}{T}t]$$



Homework Answer #8

- Problem (4a)
  - Deduce the Fourier series for the functions shown (hint: deduce the second one using superposition): Odd symmetry, only sines.



$$B_k = \frac{4}{2\pi}\int_{\alpha}^{\pi-\alpha} \sin[kt]dt$$

$$= -\frac{4}{2\pi k}\cos[kt]\,\Big|_{\alpha}^{\pi-\alpha} = \frac{2}{\pi k}\{\cos(k\alpha)-\cos(k[\pi-\alpha])\}$$

$$= \frac{2}{\pi k}\{\cos(k\alpha)-[\cos(k\pi)\cos(k\alpha)+\sin k\pi \sin k\alpha]\}$$

$$= \frac{2\cos(k\alpha)}{\pi k}\{1-\cos(k\pi)\}$$

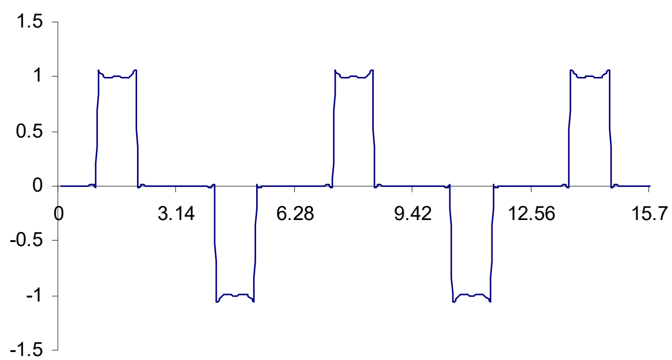
$$= \frac{2\cos(k\alpha)}{\pi k}\{1-(-1)^k\},$$

note : $1-(-1)^k = 2$ for k odd, 0 for k even

$$B_{2k-1} = \frac{4\cos([2k-1]\alpha)}{\pi[2k-1]}$$

$$A_0 = \frac{1}{2\pi}\{\int_{\alpha}^{\pi-\alpha} dt - \int_{-(\pi-\alpha)}^{-\alpha} dt\} = \frac{1}{2\pi}\{(\pi-2\alpha)-[(-\alpha)-(-\{\pi-\alpha\})]\}$$

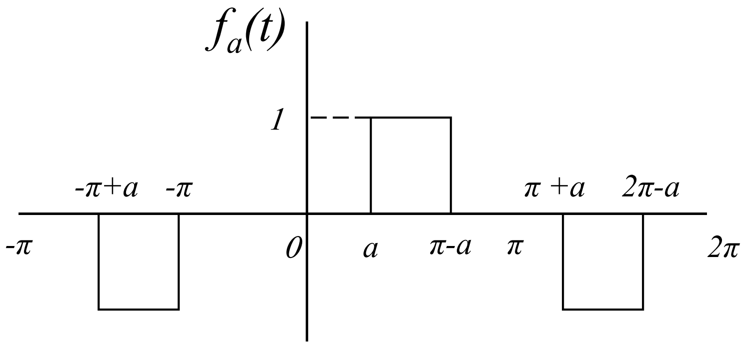
$$= 0$$

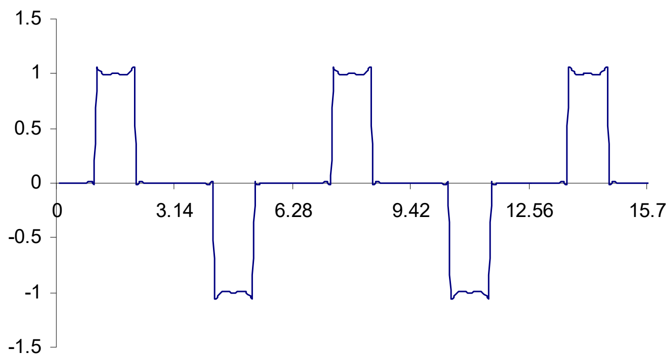


$$f(t) = \sum_{1}^{\infty} \frac{4\cos([2k-1]\alpha)}{\pi[2k-1]}\sin([2k-1]t)$$

Homework Answer #8

- Problem (4a)
  - Deduce the Fourier series for the functions shown (hint: deduce the second one using superposition): Odd symmetry, only sines.





$$a_k = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-jkt}dt = \frac{1}{2\pi}\{\int_\alpha^{\pi-\alpha} e^{-jkt}dt - \int_{\pi+\alpha}^{2\pi-\alpha} e^{-jkt}dt$$

$$= \frac{1}{-j2\pi k}(e^{-jkt}|_\alpha^{\pi-\alpha} - e^{-jkt}|_{\pi+\alpha}^{2\pi-\alpha})$$

$$= \frac{1}{-j2\pi k}(e^{-jk(\pi-\alpha)} - e^{-jk\alpha} - e^{-jk(2\pi-\alpha)} + e^{-jk(\pi+\alpha)})$$

$$= \frac{1}{-j2\pi k}\{e^{-jk\pi}[e^{-jk\alpha} + e^{jk\alpha}] - (e^{-jk\alpha} + e^{+jk\alpha})\}$$

$$= \frac{\cos(k\alpha)}{j\pi k}\{1-\cos(k\pi)\}$$

$$= \frac{\cos(k\alpha)}{j\pi k}\{1-(-1)^k\},$$

note: $1-(-1)^k = 2$ for k odd, 0 for k even

$$a_k = \frac{2\cos(k\alpha)}{j\pi k}$$

$$a_0 = \frac{1}{2\pi}\{\int_\alpha^{\pi-\alpha} dt - \int_{-(\pi-\alpha)}^{-\alpha} dt\} = \frac{1}{2\pi}\{(\pi-2\alpha)-[(-\alpha)-(-\{\pi-\alpha\})]\}$$

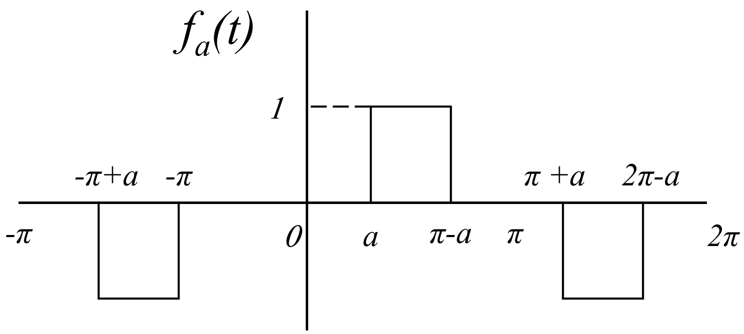
$$= 0$$

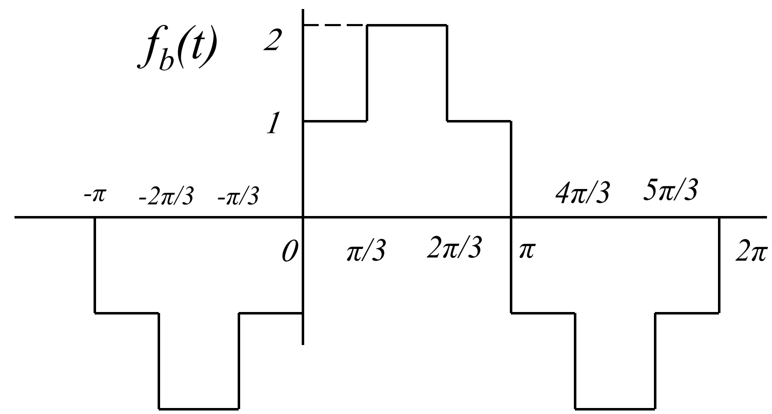
$$f(t) = \sum_1^\infty \frac{4\cos(k\alpha)}{\pi k}\cos(kt-\pi/2)$$

Homework Answer #9

Problem (4b)

– Deduce the Fourier series for the functions shown (hint: deduce the second one using superposition):





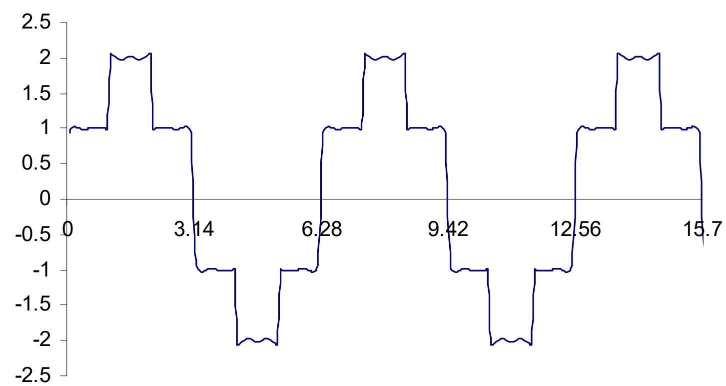
$$f_b(t) = f_a(t)|_{\alpha=\pi/3} + f_a(t)|_{\alpha=0}$$

$$f_a(t) = \sum_{1}^{\infty} \frac{4\cos([2k-1]\alpha)}{\pi[2k-1]} \sin([2k-1]t)$$

$$f_b(t) = \sum_{1}^{\infty} \frac{4\cos([2k-1]\pi/3)}{\pi[2k-1]} \sin([2k-1]t)$$

$$+ \sum_{1}^{\infty} \frac{4\cos([2k-1]\times 0)}{\pi[2k-1]} \sin([2k-1]t)$$

$$= 4\sum_{1}^{\infty} \frac{\cos([2k-1]\pi/3)+1}{\pi[2k-1]} \sin([2k-1]t)$$



5CT.7.1

$$X_k = \frac{1}{T}\int_T x(t)e^{-j\frac{2\pi k}{T}t}dt = \frac{1}{T}\int_{-T/4}^{T/4} Ae^{-j\frac{2\pi k}{T}t}dt - \frac{1}{T}\int_{T/4}^{3T/4} Ae^{-j\frac{2\pi k}{T}t}dt$$

$$= \frac{A}{T}[\int_{-T/4}^{T/4} e^{-j\frac{2\pi k}{T}t}dt - \int_{T/4}^{3T/4} e^{-j\frac{2\pi k}{T}t}dt] = \frac{A}{-j2\pi k}[e^{-j\frac{2\pi k}{T}t}|_{-T/4}^{T/4} - e^{-j\frac{2\pi k}{T}t}|_{T/4}^{3T/4}]$$

$$= \frac{A}{-j2\pi k}[e^{-j\frac{\pi k}{2}} - e^{j\frac{\pi k}{2}} - e^{-j\frac{3\pi k}{2}} + e^{-j\frac{\pi k}{2}}] = \frac{A}{-j2\pi k}[2e^{-j\frac{\pi k}{2}} - e^{j\frac{\pi k}{2}} - e^{-j\frac{3\pi k}{2}}]$$

Note: $e^{j\frac{\pi k}{2}} = e^{-j\frac{3\pi k}{2}}$

$$X_k = \frac{A}{-j2\pi k}[2e^{-j\frac{\pi k}{2}} - 2e^{j\frac{\pi k}{2}}] = \frac{2A}{\pi k}[\frac{e^{j\frac{\pi k}{2}} - e^{-j\frac{\pi k}{2}}}{2j}]$$

$$= \frac{2A}{\pi k}\sin(\frac{\pi k}{2}) = 0 \text{ for } k \text{ even}$$

$$= \frac{2A}{\pi k}\sin(\frac{\pi k}{2}) = \frac{2A}{\pi k} \text{ for } k = 1,5,9,\text{etc.} = 2l+1 \text{ for } l \text{ even} \Rightarrow l = \frac{k-1}{2}$$

$$= \frac{2A}{\pi k}\sin(\frac{\pi k}{2}) = -\frac{2A}{\pi k} \text{ for } k = 3,7,11,\text{etc.} = 2l+1 \text{ for } l \text{ odd} \Rightarrow l = \frac{k-1}{2}$$

$$X_k = \frac{2A}{\pi k}(-1)^{\frac{k-1}{2}} \text{ for } k \text{ odd}$$

5CT.7.1

Assume $x(t)$ is even

$$X_k = \frac{1}{T}\int_T x(t)e^{-j\frac{2\pi k}{T}t}dt = \int_{-T/2}^{0} x(t)e^{-j\frac{2\pi k}{T}t}dt + \int_{0}^{T/2} x(t)e^{-j\frac{2\pi k}{T}t}dt$$

In first integral substitute $t=-s; dt=-ds$;

and limits of $s: T/2 \Leftrightarrow 0$;

$$= -\int_{T/2}^{0} x(-s)e^{j\frac{2\pi k}{T}s}ds + \int_{0}^{T/2} x(t)e^{-j\frac{2\pi k}{T}t}dt$$

$$= \int_{0}^{T/2} x(s)e^{j\frac{2\pi k}{T}s}ds + \int_{0}^{T/2} x(t)e^{-j\frac{2\pi k}{T}t}dt;$$ change the limits

and use the fact the $x(t)$ is even

Now let $s=t$

$$= \int_{0}^{T/2} x(t)e^{j\frac{2\pi k}{T}t}dt + \int_{0}^{T/2} x(t)e^{-j\frac{2\pi k}{T}t}dt$$

$$= \int_{0}^{T/2} x(t)[e^{j\frac{2\pi k}{T}t} + e^{-j\frac{2\pi k}{T}t}]dt$$

$$= 2\int_{0}^{T/2} x(t)\cos[\frac{2\pi k}{T}t]dt$$

Our Problem

$$= 2\int_{0}^{T/4} \frac{A}{T}\cos[\frac{2\pi k}{T}t]dt - 2\int_{T/4}^{T/2} \frac{A}{T}\cos[\frac{2\pi k}{T}t]dt$$

$$= 2\frac{A}{T}\{\int_{0}^{T/4} \cos[\frac{2\pi k}{T}t]dt - \int_{T/4}^{T/2} \cos[\frac{2\pi k}{T}t]dt$$

$$= 2\frac{A}{T}\frac{1}{\frac{2\pi k}{T}}\{\sin[\frac{2\pi k}{T}t]|_0^{T/4} - \sin[\frac{2\pi k}{T}t]|_{T/4}^{T/2}\}$$

$$= \frac{A}{\pi k}\{\sin[\frac{2\pi k}{T}T/4] - 0 - \sin[\frac{2\pi k}{T}T/2] + \sin[\frac{2\pi k}{T}T/4]\}$$

$$= \frac{2A}{\pi k}\{\sin[\frac{\pi k}{2}]\}$$

$= 0$; for k=even

$= \frac{2A}{\pi k}$; for k=1,5,9,....=$2l+1$ for l even $\Rightarrow l = \frac{k-1}{2}$

=-$\frac{2A}{\pi k}$; for k=3,7,11,....=$2l+1$ for l odd $\Rightarrow l = \frac{k-1}{2}$

$X_k = \frac{2A}{\pi k}(-1)^{\frac{k-1}{2}}$ for k odd

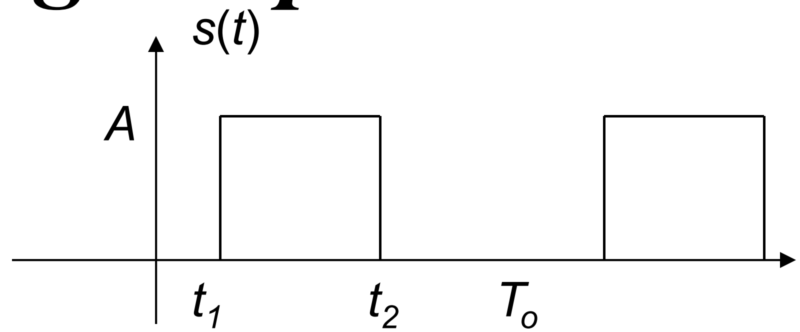
Spectrum of Biomedical Signals

Lecture 8a

Homework

- Calculate the "periodic" ECG model Fourier coefficients for a square wave model
- Use superposition to get the Fourier coefficients for the complete ECG.
  - Using Matlab with its integration function, calculate and plot the spectrum for both the square wave and triangular wave approximation of a periodic ECG. Plot the spectrum in Hz and then plot the ECG time signal using the Fourier spectral coefficients. Submit your code.

Fourier Series of a ECG Using a Square Wave Model



$$s(t) = \begin{cases} A & \text{for } t_1 \le t \le t_2 \\ 0 & \text{elsewhere} \end{cases}$$

Fourier Series of a ECG Using a Square Wave Model

$$a_k = \frac{1}{T_o}\int_0^{T_o} s(t)e^{-j(\frac{2\pi}{T_o})kt}dt = \frac{1}{T_o}\int_{t_1}^{t_2} Ae^{-j(\frac{2\pi}{T_o})kt}dt = \frac{A}{T_o}\frac{e^{-j(\frac{2\pi}{T_o})kt}}{-j(\frac{2\pi}{T_o})k}\Big|_{t_1}^{t_2} = \frac{A}{T_o}\frac{e^{-j(\frac{2\pi}{T_o})kt_2} - e^{-j(\frac{2\pi}{T_o})kt_1}}{-j(\frac{2\pi}{T_o})k}$$

$$= \frac{A}{T_o}e^{-j(\frac{2\pi}{T_o})k(\frac{t_2+t_1}{2})}\frac{e^{-j(\frac{2\pi}{T_o})k(\frac{t_2-t_1}{2})} - e^{+j(\frac{2\pi}{T_o})k(\frac{t_2-t_1}{2})}}{-j(\frac{2\pi}{T_o})k} = \frac{A}{T_o}(\frac{t_2-t_1}{2})e^{-j(\frac{2\pi}{T_o})k(\frac{t_2+t_1}{2})}\frac{e^{j(\frac{2\pi}{T_o})k(\frac{t_2-t_1}{2})} - e^{-j(\frac{2\pi}{T_o})k(\frac{t_2-t_1}{2})}}{2j(\frac{\pi}{T_o})k(\frac{t_2-t_1}{2})}$$

$$= \frac{A}{T_o}(t_2-t_1)e^{-j(\frac{2\pi}{T_o})k(\frac{t_2+t_1}{2})}\frac{\sin[(\frac{2\pi}{T_o})k(\frac{t_2-t_1}{2})]}{(\frac{2\pi}{T_o})k(\frac{t_2-t_1}{2})} = \frac{A}{T_o}(t_2-t_1)e^{-j(\frac{2\pi}{T_o})k(\frac{t_2+t_1}{2})}\mathrm{sinc}[(\frac{2\pi}{T_o})k(\frac{t_2-t_1}{2})]$$

$$= A(\frac{t_2-t_1}{T_o})\mathrm{sinc}[(\frac{2\pi}{T_o})k(\frac{t_2-t_1}{2})]e^{-j(\frac{2\pi}{T_o})k(\frac{t_2+t_1}{2})}$$

$$\Re e(a_k) = A(\frac{t_2-t_1}{T_o})\mathrm{sinc}[(\frac{2\pi}{T_o})k(\frac{t_2-t_1}{2})]\cos[-j(\frac{2\pi}{T_o})k(\frac{t_2+t_1}{2})]$$

$$\Im m(a_k) = A(\frac{t_2-t_1}{T_o})\mathrm{sinc}[(\frac{2\pi}{T_o})k(\frac{t_2-t_1}{2})]\sin[-j(\frac{2\pi}{T_o})k(\frac{t_2+t_1}{2})]$$

Fourier Series of a ECG Using a Square Wave Model

By superposition, we have for each wave

$$a_k = a_k^P + a_k^Q + a_k^R + a_k^S + a_k^T$$

$$\Re e[a_k] = \Re e[a_k^P] + \Re e[a_k^Q] + \Re e[a_k^R] + \Re e[a_k^S] + \Re e[a_k^T]$$

$$\Im m[a_k] = \Im m[a_k^P] + \Im m[a_k^Q] + \Im m[a_k^R] + \Im m[a_k^S] + \Im m[a_k^T]$$

$$a_k = \sqrt{\Re e[a_k]^2 + \Im m[a_k]^2}\, e^{j\tan^{-1}\left(\frac{\Im m[a_k]}{\Re e[a_k]}\right)}$$

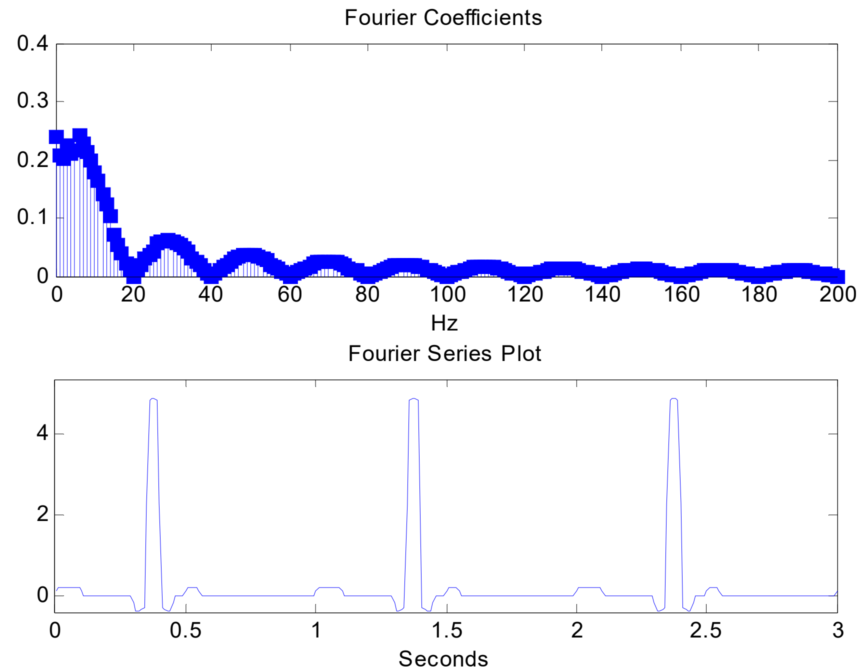
where for the *Mth* wave

$$\Re e[a_k^M] = A^M\left(\frac{t_2^M - t_1^M}{T_o}\right)\mathrm{sinc}\left[\left(\frac{2\pi}{T_o}\right)k\left(\frac{t_2^M - t_1^M}{2}\right)\right]\cos\left[-j\left(\frac{2\pi}{T_o}\right)k\left(\frac{t_2^M + t_1^M}{2}\right)\right]$$

$$\Im m[a_k] = A^M\left(\frac{t_2^M - t_1^M}{T_o}\right)\mathrm{sinc}\left[\left(\frac{2\pi}{T_o}\right)k\left(\frac{t_2^M - t_1^M}{2}\right)\right]\sin\left[-j\left(\frac{2\pi}{T_o}\right)k\left(\frac{t_2^M + t_1^M}{2}\right)\right]$$

Square Wave

```
clear all;
starttime=clock
T=1;kmax=200;
A=.2;tlower=0;tupper=.1;
p=fourierint(A,kmax,T,tlower,tupper);
A=-.4;tlower=.3;tupper=.35;
q=fourierint(A,kmax,T,tlower,tupper);
A=5;tlower=.35;tupper=.4;
r=fourierint(A,kmax,T,tlower,tupper);
A=-.4;tlower=.4;tupper=.45;
s=fourierint(A,kmax,T,tlower,tupper);
A=.2;tlower=.5;tupper=.55;
t=fourierint(A,kmax,T,tlower,tupper);
a=p+q+r+s+t;
k=(0:length(a)-1);
subplot(2,1,1);
stem(k/T,abs(a),'Marker','square','MarkerFaceColor','b')
title('Fourier Coefficients');
xlabel('Hz');
time=(0:T/100:3*T);
maxtime=length(time);
for i=1:maxtime;
   x=a(1);
   for j=2:kmax;
      if a(j)~=0
        x=x+2*abs(a(j))*cos(2*pi*(j-1)/T*time(i)+atan2(imag(a(j)),real(a(j))));
      end
   end
   y(i)=x;
end
subplot(2,1,2)
plot(time,y);
title('Fourier Series Plot');
xlabel('Seconds');
axis([ time(1) max(time) min(y)*1.1 max(y)*1.1]);
endtime=clock
hours=endtime(4)-starttime(4)
```

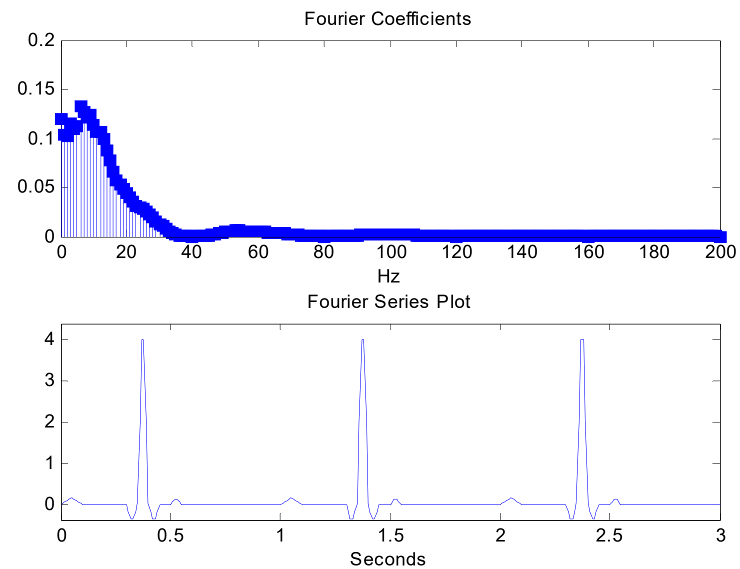


Square Wave

```
function  a=fourierint(x,maxk,T,tlower,tupper)
syms t
k=0;
basis=x*exp(i*(-2*pi*k*t/T));
a0=int(basis,tlower,tupper);
for k=1:maxk
   basis=x*exp(i*(-2*pi*t*k/T));
   b(k)=int(basis,t,tlower,tupper);
end
a=double([a0 b]);
```

Triangular Wave

```
clear all;
starttime=clock
T=1;kmax=200;
A=.2;tlower=0;tupper=.1;tmid=(tupper+tlower)/2;
p=fourierintramp(A,kmax,T,tlower,tmid,tupper);
A=-.4;tlower=.3;tupper=.35;tmid=(tupper+tlower)/2;
q=fourierintramp(A,kmax,T,tlower,tmid,tupper);
A=5;tlower=.35;tupper=.4;tmid=(tupper+tlower)/2;
r=fourierintramp(A,kmax,T,tlower,tmid,tupper);
A=-.4;tlower=.4;tupper=.45;tmid=(tupper+tlower)/2;
s=fourierintramp(A,kmax,T,tlower,tmid,tupper);
A=.2;tlower=.5;tupper=.55;tmid=(tupper+tlower)/2;
t=fourierintramp(A,kmax,T,tlower,tmid,tupper);
a=p+q+r+s+t;
k=(0:length(a)-1);
subplot(2,1,1);
stem(k/T,abs(a),'Marker','square','MarkerFaceColor','b')
title('Fourier Coefficients');
xlabel('Hz');
time=(0:T/100:3*T);
maxtime=length(time);
for i=1:maxtime;
   x=a(1);
   for j=2:kmax;
      if a(j)~=0
         x=x+2*abs(a(j))*cos(2*pi*(j-1)/T*time(i)+atan2(imag(a(j)),real(a(j))));
      end
   end
   y(i)=x;
end
subplot(2,1,2)
plot(time,y);
title('Fourier Series Plot');
xlabel('Seconds');
axis([ time(1) max(time) min(y)*1.1 max(y)*1.1]);
endtime=clock
hours=endtime(4)-starttime(4)
```



Triangular Wave

```
function  a=fourierintramp(x,maxk,T,tlower,tmid,tupper)
syms t
k=0;
basispos=x*(t-tlower)/(tmid-tlower)*exp(i*(-2*pi*k*t/T));
basisneg=-x*(t-tupper)/(tupper-tmid)*exp(i*(-2*pi*k*t/T));
a0pos=int(basispos,tlower,tmid);
a0neg=int(basisneg,tmid,tupper);
a0=a0pos+a0neg;
for k=1:maxk
   basis=x*(t-tlower)/(tmid-tlower)*exp(i*(-2*pi*t*k/T));
   bpos(k)=int(basis,t,tlower,tmid);
   basisneg=-x*(t-tupper)/(tupper-tmid)*exp(i*(-2*pi*k*t/T));
   bneg(k)=int(basisneg,tmid,tupper);
   b(k)=bpos(k)+bneg(k);
end
a=double([a0 b]);
```